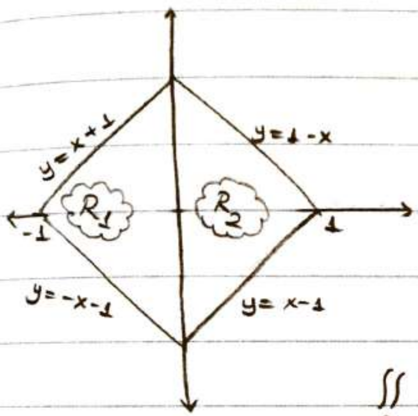
(15.2 → 95)

ex! $\iint_R (y - 2x^2)\, dA$ where R is the region bounded by the square $|x| + |y| = 1$



$R_1 : -1 \le x \le 0$

$-x-1 \le y \le x+1$

$R_2 : 0 \le x \le 1$

$x-1 \le y \le 1-x$

$$\iint_R (y-2x^2)\,dA = \iint_{R_1} (y-2x^2)\,dA + \iint_{R_2} (y-2x^2)\,dA$$

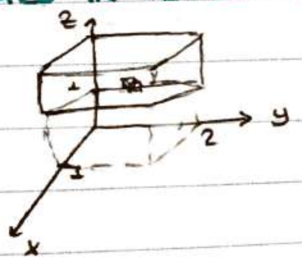
$$= \int_{-1}^{0}\int_{-x-1}^{x+1} (y-2x^2)\,dy\,dx + \int_{0}^{1}\int_{x-1}^{1-x} (y-2x^2)\,dy\,dx$$

15.5 → MULTIPLE INTEGRALS IN TRIPLE COORDINATES

$0 \le x \le 1$

$0 \le y \le 2$

$1 \le z \le 3$



$f(x,y,z) \rightarrow$ density

$$\text{Mass} \approx \sum_i \sum_j \sum_k f(x_i, y_j, z_k)\, \Delta x_i\, \Delta y_j\, \Delta z_k$$

Take limit

$$\text{Mass} = \iiint_{[0,1]\times[0,2]\times[1,3]} f(x,y,z)\,dV$$

Fubini Theorem

$$= \int_{x=0}^{1}\int_{y=0}^{2}\int_{z=1}^{3} f(x,y,z)\,dz\,dy\,dx$$

→ can be write 4 more ways.

$$= \int_{y=0}^{2}\int_{x=0}^{1}\int_{z=1}^{3} f(x,y,z)\,dz\,dx\,dy$$

In general let R be the region

$a \le x \le b$

$g_1(x) \le y \le g_2(x)$

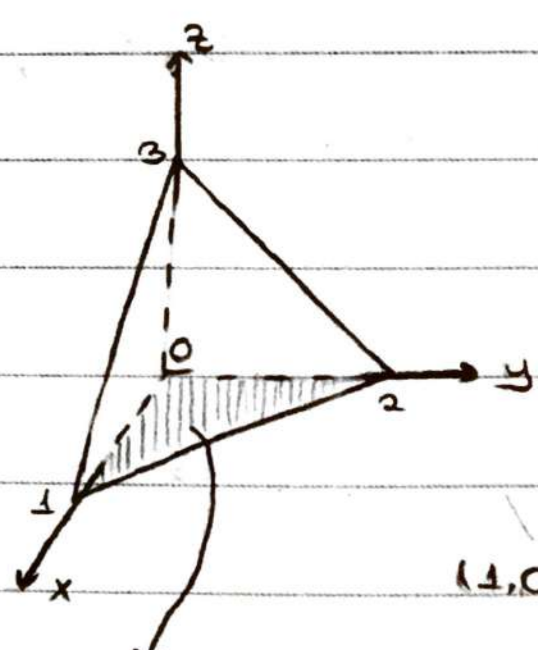
$h_1(x,y) \le z \le h_2(x,y)$

FUBINI

$$\iiint_R f(x,y,z)\,dV = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)}\int_{z=h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

$$\text{Volume}(R) = \iiint_R 1\,dV$$

ex:



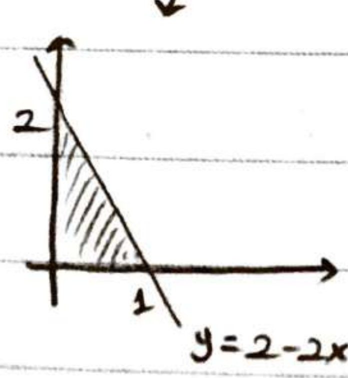
Find the volume of the tetrahedron with vertices $(0,0,0)$ $(1,0,0)$ $(0,2,0)$ $(0,0,3)$

$ax+by+cz=d$

$(1,0,0) \to a=d$

$(0,2,0) \to 2b=d$

$(0,0,3) \to 3c=d$



$dx + \frac{d}{2}y + \frac{d}{3}z = d$

$3x + \frac{3}{2}y + z = 3$

$0 \le x \le 1$

$0 \le y \le 2-2x$

$0 \le z \le 3-3x-3y/2$

$$\text{Volume} = \int_0^1 \int_0^{(2-2x)} \int_0^{(3-3x-3y/2)} 1 \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} \left(z \Big|_0^{3-3x-3y/2} \right) dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} (3-3x-3y/2)\, dy dx$$

$$= \int_0^1 \left(3-3x-3y/2 \Big|_0^{2-2x} \right) dx$$

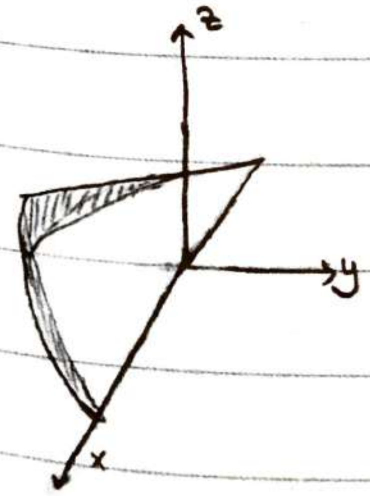
$$= \int_0^1 \left[3(2-2x) - 3x(2-2x) - \frac{3}{4}(2-2x)^2 \right] dx$$

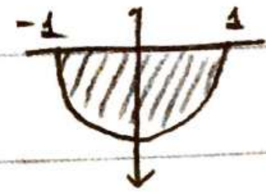
$$= \int_0^1 (3-6x+3x^2)\, dx$$

$$= 3x-3x^2+x^3 \Big|_0^1 = 1 \nearrow \left(\frac{abc}{6}\right)$$

(book 26)

ex: Find the volume of the wedge cut from the cylinder $x^2+y^2=1$, by the planes $z=-y$ and $z=0$





$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le 0$

$0 \le z \le -y$

First Method

$$\text{Volume} = \int_{x=-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{z=0}^{-y} dz\,dy\,dx$$

$$= \int_{-1}^{1} \int_{(-\sqrt{1-x^2}}^{0} (-y)\,dy\,dx$$

$$= \int_{-1}^{1} \frac{-y^2}{2} \Big|_{-\sqrt{1-x^2}}^{0} dx$$

$$= \int_{-1}^{1} \frac{1-x^2}{2}\,dx = \frac{1}{2}\left(x-\frac{x}{3}\right)^2 \Big|_{-1}^{1}$$

$$= \frac{1}{2}\left(1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right) = \frac{2}{3}$$

Second Method

$$\text{Volume} = \int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} dz\,dy\,dx$$

$$= \int_{-1}^{0} \int_{-\sqrt{1-x^2}}^{0} (-y)\,dy\,dx$$

$$\pi \leq \theta \leq 2\pi$$

$$0 \leq r \leq 1$$

$$= \int_{\theta=\pi}^{2\pi} \int_{0}^{1} -r\sin\theta \; r\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi} \frac{-r^3}{3} \sin\theta \Big|_{0}^{1} d\theta$$

$$= -\frac{1}{3} \cdot \int_{\pi}^{2\pi} \sin\theta\,d\theta$$

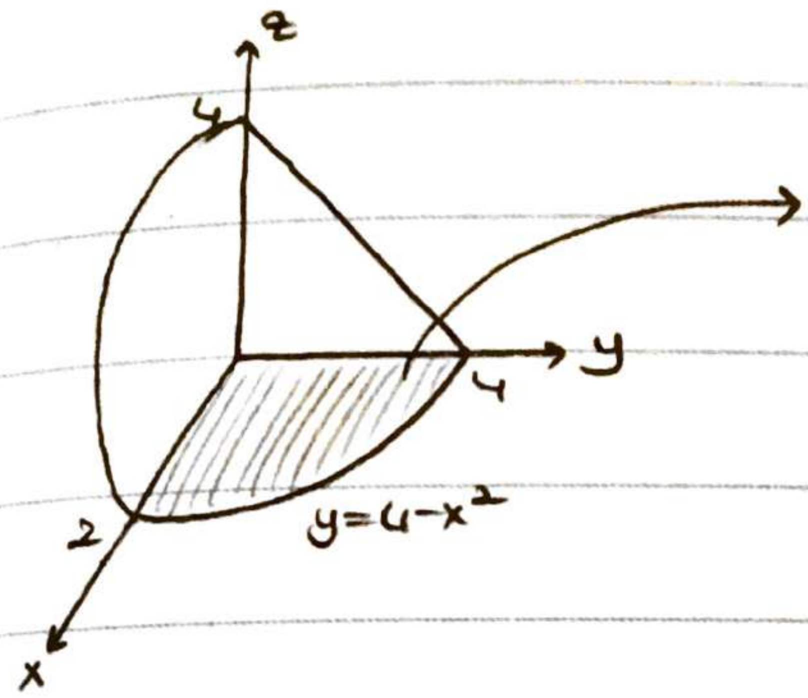
$$= -\frac{1}{3}(-\cos\theta) \Big|_{\pi}^{2\pi}$$

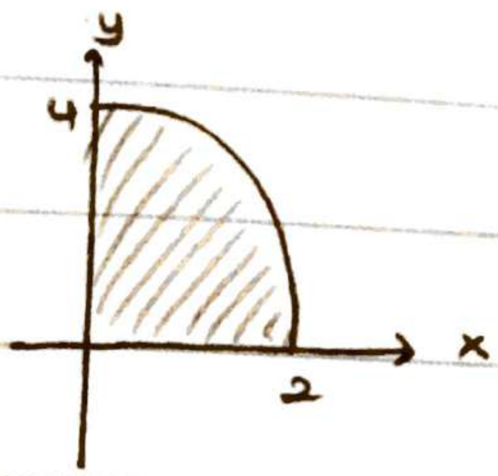
$$= \frac{1}{3}(1+1) = 2/3$$

(book 30)

ex! Find the volume. The region in the first octant bounded by the coordinate planes and the surface

$z = 4 - x^2 - y$





$0 \leq x \leq 2$

$0 \leq y \leq 4 - x^2$

$0 \leq z \leq 4 - x^2 - y$

$$\text{Volume} = \int_0^2 \int_0^{(4-x^2)} \int_0^{(4-x^2-y)} 1 \, dz\, dy\, dx$$

15.6 → MOMENTS AND CENTERS OF MASS

3D SOLID

$f(x,y,z) \rightarrow$ density function

$$\text{Mass} = \iiint_D f(x,y,z)\, dV$$

* First moments about the coordinate planes

$$M_{yz} = \iiint_D x \, \rho(x,y,z)$$

$$M_{xz} = \iiint_D y \, \rho(x,y,z)$$

$$M_{xy} = \iiint_D z \, \rho(x,y,z)$$